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The rolling tachyon as a matrix model

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Abstract: We express all correlation functions in timelike boundary Liouville theory as unitary matrix integrals and develop efficient techniques to evaluate these integrals. We compute large classes of correlation functions explicitly, including an infinite number of terms in the boundary state of the rolling tachyon. The matrix integrals arising here also determine the correlation functions of gauge invariant operators in two dimensional Yang-Mills theory, suggesting an equivalence between the rolling tachyon and QCD_2.

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1. Introduction

The decay of unstable D–brane systems is a simple example of a time dependent background in which one would like to understand the behaviour of string theory. More generally, the study of time dependent backgrounds is of interest for the simple reason that we appear to live in one. Unfortunately this is a notoriously difficult issue to even formulate in a clean way. It is therefore of no small interest that the rolling tachyon backgrounds of Sen [1] are described in terms of exactly soluble boundary conformal field theories. These backgrounds are therefore tractable, and one may hope that lessons learned here carry over to more general time dependent situations.

The time dependent background studied in this paper is the rolling tachyon corresponding to the decay of a D25 brane in bosonic string theory. We will be specifically interested in the case where the tachyon on the worldvolume of the D25-brane sits at the top of its potential at $t = -\infty$ and rolls to the minimum as $t \to +\infty$ [2]–[5]. In terms of conformal field theory this is described by the usual $c = 25$ worldsheet theory for the spatial fluctuations of the open string plus the action

$$-\frac{1}{2\pi}\int_{\Sigma}\partial X^0\bar{\partial}X^0 + g\int_{\partial\Sigma} e^{X^0(e^{ix^0})}\,, \tag{1.1}$$

for the temporal fields. We will refer to this theory of a negative norm boson with an exponential boundary interaction as Timelike Boundary Liouville theory and resist the terminology 1/2 S-brane. Many other interesting avenues of investigation into rolling tachyons have been pursued in the recent literature [6]–[13].

This paper will focus on general classes of correlation functions in Timelike Boundary Liouville theory (TBL). The main result of the investigation is a demonstration that all correlators in this theory, and hence the rolling tachyon background, can be expressed as unitary matrix integrals. Moreover, these integrals permit an expansion in simple quantities from the theory of groups, such as the characters of the symmetric groups. These relations provide an efficient, purely algebraic, algorithm for computing all correlation functions of the theory. As explicit examples we obtain infinitely many coefficients in the boundary state of the rolling tachyon. We also compute m-point correlators of exponentials of the field X^0 in the background of the rolling tachyon, for bulk and boundary fields.

The appearance of matrix integrals strongly suggests that the full dynamics of this time dependent background should be captured by a unitary matrix model. The obvious question is, which matrix model? An indication of which class of matrix models we should look at comes from the fact that correlation functions of timelike boundary Liouville theory, when expressed in terms of $\mathrm{U}(n)$ matrices, are easily recognizable as correlation functions of gauge invariant operators in two-dimensional Yang-Mills theory [15, 16]. In particular, the one point functions which lead to the boundary state coefficients are precisely the same as the correlation functions between pairs of Wilson loops in QCD_2, in the limit of small separations. We are thus led to conjecture that the timelike boundary Liouville theory is the same theory as QCD_2. As it is widely believed that two dimensional Yang-Mills can itself be formulated as a matrix model, the same will be true for TBL. The matrix model description would amount to a holographic projection of TBL to one dimension less. The surprising twist to the story is that the holographic description itself is the gauge fixed version of QCD_2, i.e. a theory in one dimension *more*. It would clearly be interesting to develop these relations further. Recently other relations between matrix models and tachyons have been proposed in [14].

The remainder of the paper is organized as follows. In section two we review the basic observation of [3] that led to our investigation, namely, that the partition function of the rolling tachyon background is related to the Haar measure of the unitary group. We also review how S_N, the symmetric group of N objects, encodes the spectrum of the closed bosonic string. In section three we show that bulk one point functions of massive closed string states are in fact matrix integrals. We establish our technique for evaluating these integrals and obtain general expressions for infinitely many boundary state coefficients. In section four we consider bulk m-point functions, demonstrating how these may also be written as matrix integrals and readily evaluated. These results are extended to boundary

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correlators in section five. In section six we discuss our conjecture relating timelike boundary Liouville theory to two dimensional gauge theory and hence a matrix model. Finally, in an attempt at making the paper somewhat self-contained, we have included several appendices reviewing some facts from the representation theory of the unitary groups, the characters of the symmetric group, and the theory of symmetric functions.

2. Preliminaries

In this section we review a few of the key ingredients needed in this paper. First, the representation of correlators in the rolling tachyon background as integrals over $\mathrm{U}(n)$ group manifolds; and then the connection between string states and conjugacy classes of the symmetric group.

2.1 Correlators as $\mathrm{U}(n)$ integrals

The basic observables in the rolling tachyon background are the correlation functions of vertex operators, corresponding to open and closed string states. A general correlator is of the form

$$\mathcal{A}^{(\Pi V_i)} = \left\langle \prod_i V_i(z,\bar{z}) \right\rangle_{TBL} = \left\langle \prod_i V_i(z,\bar{z}) e^{-I_{\text{bndy}}} \right\rangle , \tag{2.1}$$

where $z, \bar{z}$ are coordinates on the unit disk and the boundary interaction is that of the rolling tachyon

$$I_{\text{bndy}} = g \int dt \, \exp\left(X^0(t)\right) . \tag{2.2}$$

One approach to evaluating these expressions is to treat the boundary interaction perturbatively and write

$$\mathcal{A}^{(\Pi V_i)} = \sum_{n=0}^{\infty} \frac{(-2\pi g e^{x^0})^n}{n!} \left\langle \prod_i V_i(z,\bar{z}) \int \prod_{i=1}^{n} \frac{dt_i}{2\pi} e^{\hat{X}^0(e^{it_i})} \right\rangle , \tag{2.3}$$

where the field X^0 was divided into a zeromode x^0 and a fluctuating part $\hat{X}^0$. We will leave the zero-mode x^0 unintegrated, a standard procedure when interpreting the CFT in spacetime as a rolling tachyon. It is useful to introduce a separate notation for the nth order contribution to the correlator eq. (2.3)

$$\mathcal{A}_n^{(\Pi V_i)} \equiv \frac{1}{n!} \left\langle \prod_i V_i(z,\bar{z}) \int \prod_{i=1}^{n} \frac{dt_i}{2\pi} e^{\hat{X}^0(e^{it_i})} \right\rangle , \tag{2.4}$$

and so write

$$\mathcal{A}^{(\Pi V_i)} = \sum_{n=0}^{\infty} (-\tilde{g})^n \mathcal{A}_n^{(\Pi V_i)} , \tag{2.5}$$

where $\tilde{g} \equiv 2\pi g e^{x^0}$.

The key observation for the techniques developed in this paper is that the contractions that do not involve the vertex operators take the form

$$\left\langle \prod_{i=1}^{n} e^{\hat{X}^0(e^{it_i})} \right\rangle = \prod_{i<j} |e^{it_i} - e^{it_j}|^2 = \prod_{i<j} 4\sin^2\left(\frac{t_i - t_j}{2}\right) \equiv \Delta^2(t)\,, \tag{2.6}$$

where $\Delta(t)$ is the Vandermonde determinant for the group $\mathrm{U}(n)$ [3]. It follows that the disk amplitude with no vertex operator insertions becomes

$$\mathcal{A}_n^{\mathrm{vac}} = \frac{1}{n!}\int \left\langle \prod_{i=1}^{\infty} \frac{dt_i}{2\pi} e^{\hat{X}^0(e^{it_i})} \right\rangle = \frac{1}{n!}\int \prod_{i=1}^{n} \frac{dt_i}{2\pi} \Delta^2(t) = \frac{1}{\mathrm{vol}(\mathrm{U}(n))}\int dU = 1\,, \tag{2.7}$$

where dU is the Haar measure for $\mathrm{U}(n)$. Summing over all orders in the perturbation series, we find

$$\mathcal{A}^{\mathrm{vac}} = \sum_{n=0}^{\infty} (-\tilde{g})^n = \frac{1}{1+\tilde{g}} \equiv f(x^0)\,. \tag{2.8}$$

The function f is the partition function of the theory, treated as a function of the unintegrated zero-mode.

The appearance of the Vandermonde determinant in the vacuum amplitude begs the question of whether other correlation functions, with the $V_i(z,\bar{z})$ retained, can be similarly represented in terms of unitary matrices. We will see that this is indeed the case.

2.2 Closed strings and the symmetric group

The mass spectrum of closed bosonic string theory is given by

$$\frac{1}{2}m^2 + 2 = \sum_n N_n n + \sum_{\tilde{n}} \tilde{N}_{\tilde{n}} \tilde{n} \equiv N + \tilde{N}\,, \tag{2.9}$$

in units with $\alpha' = 1$. The level matching condition states that $N = \tilde{N}$, while the N_n and $\tilde{N}_{\tilde{n}}$ need not be related. The spectrum at a given mass level is thus labelled by a pair of partitions of the integer N. For example, if $\frac{1}{2}m^2 + 2 = 3 + 3$ the possible string states are identified by pairs of partitions of 3. The partitions are $(3), (1,2), (1,1,1)$ in this case. The state

$$\alpha_{-3}\tilde{\alpha}_{-2}\tilde{\alpha}_{-1}|0\rangle\,, \tag{2.10}$$

corresponds to the partition (3) for the left movers and the partition $(1,2)$ for the right movers. The states for other pairs of partitions are readily written down as well.

The utility in taking this point of view is that the partitions of N are in a one to one correspondence with the conjugacy classes of the symmetric group S_N. So we may just as well view the oscillator structure of each 'side' of the string as being labelled by a conjugacy class of S_N. In the example above we are dealing with S_3 and the state written in eq. (2.10) corresponds to choosing the conjugacy class of long cycles (123) for the left movers and the conjugacy class of $(12)(3)$ for the right movers. Here we are employing the standard notation for elements of S_3. In general, the partition $N = N_1\nu_1 + N_2\nu_2 + \cdots N_k\nu_k$, where the N_i are the multiplicities of the ν_i, corresponds to the oscillators

$$\alpha_{-\nu_1}^{N_1}\alpha_{-\nu_2}^{N_2}\cdots\alpha_{-\nu_k}^{N_k}\,. \tag{2.11}$$

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This set of oscillators is thus labelled by the conjugacy class of S_N which contains N_1 cycles of length ν_1, N_2 cycles of length ν_2 and so on. We will denote this conjugacy class $\sigma = (\nu_1^{N_1}, \nu_2^{N_2}, \ldots, \nu_k^{N_k})$. Here and in the folowing all oscillators α_{ν_i} will be assumed to be temporal unless indicated otherwise.

A general massive string state at level $N = \frac{1}{4}m^2 + 1$ can be expressed in oscillator notation as

$$\mathcal{N}^{(\sigma;\tilde{\sigma})} \alpha_{-\nu_1}^{N_1} \alpha_{-\nu_2}^{N_2} \cdots \alpha_{-\nu_k}^{N_k} \tilde{\alpha}_{-\tilde{\nu}_1}^{\tilde{N}_1} \tilde{\alpha}_{-\tilde{\nu}_2}^{\tilde{N}_2} \cdots \tilde{\alpha}_{-\tilde{\nu}_{\tilde{k}}}^{\tilde{N}_{\tilde{k}}} |0,0\rangle \equiv |\sigma, \tilde{\sigma}\rangle \,, \tag{2.12}$$

where $\sigma, \tilde{\sigma}$ denote the conjugacy classes of S_N which label the state we are considering and

$$\mathcal{N}^{(\sigma;\tilde{\sigma})} = \left[\prod_i \nu_i^{N_i} N_i! \prod_{\tilde{i}} \tilde{\nu}_{\tilde{i}}^{\tilde{N}_{\tilde{i}}} \tilde{N}_{\tilde{i}}! \right]^{-\frac{1}{2}} , \tag{2.13}$$

is the normalization factor. The corresponding vertex operators are given by

$$V^{(\sigma;\tilde{\sigma})}(z, \bar{z}) = \mathcal{N}^{(\sigma;\tilde{\sigma})} \prod_{i=1}^{k} \left(\frac{\sqrt{2}}{(\nu_i - 1)!} \partial^{\nu_i} X^0(z, \bar{z}) \right)^{N_i} \prod_{\tilde{i}=1}^{\tilde{k}} \left(\frac{\sqrt{2}}{(\tilde{\nu}_{\tilde{i}} - 1)!} \bar{\partial}^{\tilde{\nu}_{\tilde{i}}} X^0(z, \bar{z}) \right)^{\tilde{N}_{\tilde{i}}} , \tag{2.14}$$

where the numerical constants were determined using the operator-state correspondence (as in [17]).

3. Couplings to closed strings

In this section we introduce the matrix method by considering the simplest correlators, the one point functions of closed strings in the rolling tachyon background. We also point out a suggestive connection to QCD_2. Finally, we relate our results to those obtained using boundary state methods.

3.1 Correlators as matrix integrals

The nth order contribution to the disk amplitude with a single closed string inserted at the origin, can be written as

$$\mathcal{A}_n^{(\sigma,\tilde{\sigma})} = \int \prod_{i=1}^{n} \frac{dt_i}{2\pi} A_n^{(\sigma;\tilde{\sigma})} = \int \prod_{i=1}^{n} \frac{dt_i}{2\pi} \left\langle {}^{\circ}_{\circ} V^{(\sigma;\tilde{\sigma})}(0,0) {}^{\circ}_{\circ} \prod_{l=1}^{n} \exp(X^0(w_l)) \right\rangle , \tag{3.1}$$

where the $w_l = e^{it_l}$ represent the positions of the tachyon vertex operators associated with the rolling tachyon background, and the notation ${}^{\circ}_{\circ}\ {}^{\circ}_{\circ}$ denotes boundary normal ordering. The Green's function on the unit disk is

$$G_l \equiv G(z, w_l) = \log|z - w_l| + \log|z\bar{w}_l - 1| \,, \tag{3.2}$$

for a field with temporal signature. Carrying out contractions involving the closed string vertex we find

$$A_n^{(\sigma;\tilde{\sigma})} = \mathcal{N}^{(\sigma;\tilde{\sigma})} \prod_{i=1}^{k} \left[\frac{\sqrt{2}}{(\nu_i - 1)!} \sum_{l=1}^{n} \partial^{\nu_i} G_l \right]^{N_i} \prod_{\tilde{i}=1}^{\tilde{k}} \left[\frac{\sqrt{2}}{(\tilde{\nu}_{\tilde{i}} - 1)!} \sum_{\tilde{l}=1}^{n} \bar{\partial}^{\tilde{\nu}_{\tilde{i}}} G_{\tilde{l}} \right]^{\tilde{N}_{\tilde{i}}} \left\langle \prod_{j=1}^{n} \exp(X^0(w_j)) \right\rangle . \tag{3.3}$$

It is straightforward to show that

$$\partial^{\nu_i} G_l = -(\nu_i - 1)! \exp(-i\nu_i t_l) \,, \tag{3.4}$$

where we have taken $z = 0$ and $w_l = e^{it_l}$. The factor $(\nu_i - 1)!$ cancels a similar factor in the vertex operator normalization; and so, after using eq. (2.6), we find

$$A_n^{(\sigma;\tilde{\sigma})} = \mathcal{N}'^{(\sigma;\tilde{\sigma})}\, \Delta(t)^2 \prod_{i=1}^{k} \left[\sum_{l=1}^{n} e^{-it_l \nu_i}\right]^{N_i} \prod_{\tilde{i}=1}^{\tilde{k}} \left[\sum_{\tilde{l}=1}^{n} e^{it_{\tilde{l}} \tilde{\nu}_{\tilde{i}}}\right]^{\tilde{N}_{\tilde{i}}} \,, \tag{3.5}$$

where

$$\mathcal{N}'^{(\sigma;\tilde{\sigma})} = \left[\prod_i \left(\frac{\nu_i}{2}\right)^{N_i} N_i! \prod_{\tilde{i}} \left(\frac{\tilde{\nu}_{\tilde{i}}}{2}\right)^{\tilde{N}_{\tilde{i}}} \tilde{N}_{\tilde{i}}!\right]^{-\frac{1}{2}} . \tag{3.6}$$

This generalizes the results of [3] to the case of the most general massive closed string.

Our goal is to represent the amplitude eq. (3.1) as a matrix integral. A $\mathrm{U}(n)$ matrix U can always be written as $U = \exp(iT)$ for some hermitean matrix T. Further the eigenvalues of U can be written as e^{it_i} where the t_i are the eigenvalues of T. In this diagonal basis we see that

$$\mathrm{Tr}(U) = \sum_{i=1}^{n} \exp(it_i) \,, \tag{3.7}$$

or more generally,

$$[\mathrm{Tr}(U^{\nu_i})]^{N_i} = \left[\sum_{l=1}^{n} \exp(it_l \nu_i)\right]^{N_i} . \tag{3.8}$$

The amplitude eq. (3.1) with integrand eq. (3.5) can therefore be written as

$$\mathcal{A}_n^{(\sigma;\tilde{\sigma})} = \mathcal{N}'^{(\sigma;\tilde{\sigma})} I_n^{(\sigma;\tilde{\sigma})} \,, \tag{3.9}$$

where

$$I_n^{(\sigma;\tilde{\sigma})} = \frac{1}{\mathrm{vol}(\mathrm{U}(n))} \int dU \prod_{i=1}^{k} [\mathrm{Tr}(U^{\nu_i})]^{N_i} \prod_{\tilde{i}=1}^{\tilde{k}} \left[\mathrm{Tr}\left(U^{\dagger \tilde{\nu}_{\tilde{i}}}\right)\right]^{\tilde{N}_{\tilde{i}}} . \tag{3.10}$$

This is a general expression for *all* one point functions of massive string modes in the rolling tachyon background. Integrals of this form can be evaluated elegantly and efficiently, by exploiting group theory methods.

Before showing how to do this it is worth noting the intriguing connection to two-dimensional $\mathrm{U}(n)$ gauge theory. Namely, the matrix integral in eq. (3.10) is *exactly* the expression one obtains in QCD_2 as the correlation function between a pair of Wilson loops in the limit where the area of the Riemann surface (target space) shrinks to zero [15, 16]. The precise connection between the rolling tachyon background and QCD_2 is a bit mysterious, since in the case of interest here each term in the perturbative expansion (2.3) corresponds to a Wilson loop correlator in a different gauge theory i.e. $\mathrm{U}(2), \mathrm{U}(3)$ and so on. We will comment more on this fascinating connection in section six.

3.2 Integrals of class functions

The tool needed to evaluate the amplitudes given in eq. (3.10) is a classic piece of mathematics known as Schur-Weyl duality, which relates the irreducible representations of $\mathrm{U}(n)$ to those of the symmetric group (See appendix A for more details).

Consider a function of the form $f(U)=\prod_{i=1}^{k}\left[\mathrm{Tr}(U^{\nu_i})\right]^{N_i}$, with U in the defining $n\times n$ representation of $\mathrm{U}(n)$. $f(U)$ is invariant under conjugation of U by elements of $\mathrm{U}(n)$; and so depends only on the conjugacy classes of U. Functions with this property are known as class functions. The integrand in eq. (3.10) is the product of class functions on $\mathrm{U}(n)$ and this makes the evaluation of the integral extremely easy. This is because the characters $\chi_\lambda(U)$ of the irreducible representations of $\mathrm{U}(n)$ provide an orthonormal basis for class functions. In the present case $f(U)$ can be expanded in this basis as

$$f(U)=\prod_{i=1}^{k}[\mathrm{Tr}(U^{\nu_i})]^{N_i}=\sum_{\lambda\leq n}\chi_\lambda(\sigma)\chi_\lambda(U)\,, \tag{3.11}$$

where the summation index λ refers to irreducible representations of the symmetric group S_N; and $\chi_\lambda(\sigma)$ is the character of $\sigma\in S_N$ in the irreducible representation λ. Recall that σ is a representative of the conjugacy class which labels the left-moving side of the string state. The irreducible representations of S_N are classified by the partitions of N which, in turn, we can picture as Young frames with N boxes. Of course, a given Young frame also corresponds to an irreducible representation of $\mathrm{U}(n)$, and it is in this sense that the index λ is used in eq. (3.11) to label $\mathrm{U}(n)$ representations, as well as representations of S_N. However, in some cases Young frames with N boxes have more than n boxes in one column, and then the corresponding $\mathrm{U}(n)$ representation vanishes, by complete anti-symmetry of the corresponding tensors. This imposes an important restriction on the sum over λ in eq. (3.11). We have introduced the notation $\lambda\leq n$ as a shorthand to remember that we should sum only over S_N representations that make sense also as $\mathrm{U}(n)$ representations.

We are now ready to use the expansion eq. (3.11) to evaluate the matrix integral eq. (3.10). We find

$$\begin{aligned}
I_n^{(\sigma;\tilde\sigma)} &= \frac{1}{\mathrm{vol}(\mathrm{U}(n))}\int dU\prod_{i=1}^{k}[\mathrm{Tr}(U^{\nu_i})]^{N_i}\prod_{\tilde\imath=1}^{\tilde k}\left[\mathrm{Tr}(U^{\dagger\tilde\nu_{\tilde\imath}})\right]^{\tilde N_{\tilde\imath}}\\
&= \frac{1}{\mathrm{vol}(\mathrm{U}(n))}\sum_{\lambda\lambda'\leq n}\chi_\lambda(\sigma)\chi_{\lambda'}(\tilde\sigma)\int dU\chi_\lambda(U)\chi_{\lambda'}\left(U^\dagger\right)\\
&= \sum_{\lambda\lambda'\leq n}\chi_\lambda(\sigma)\chi_{\lambda'}(\tilde\sigma)\delta_{\lambda\lambda'}\\
&= \sum_{\lambda\leq n}\chi_\lambda(\sigma)\chi_\lambda(\tilde\sigma)\,,
\end{aligned} \tag{3.12}$$

where, in going from the second to third line, we have used the orthogonality of group characters

$$\frac{1}{\mathrm{vol}(\mathrm{U}(n))}\int dU\chi_\lambda(U)\bar\chi_{\lambda'}(U)=\delta_{\lambda\lambda'}\,, \tag{3.13}$$

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and also $\chi_\lambda(U^\dagger) = \bar{\chi}_\lambda(U)$ for unitary representations. The general amplitude thus reduces to evaluating a sum over the characters of the irreducible representations of S_N. Before considering explicit examples it is worthwhile to make a few general remarks.

When $n \geq N$ there are no Young frames with N boxes that have more than n columns; so, in this case, the final sum in eq. (3.12) is over all irreducible representations of S_N and is given by the completeness relation[1]

$$I_n^{(\sigma;\tilde{\sigma})} = \sum_\lambda \chi_\lambda(\sigma)\chi_\lambda(\tilde{\sigma}) = \delta_{\sigma\tilde{\sigma}} \prod_k \nu_k^{N_k} N_k! \,; \qquad n \geq N \,, \tag{3.14}$$

where the integers ν_k, N_k are those defining the conjugacy class σ.

When $n < N$ the sum in eq. (3.11) does not run over all irreducible representations of S_N and there is no simple expression analogous to eq. (3.14). In particular, characters of different conjugacy classes are not in general orthogonal with the restricted sum. In these cases the final summation must be performed by directly evaluating the characters of the representations which do appear. For all the states at level N what we need is simply the character table of S_N, a standard quantity which can be computed using a variety of techniques. In appendix B we outline one such technique, known as the Murnaghan-Nakayama Rule.

To compute the amplitude for a given closed string state we want to sum over *all* orders in perturbation theory, i.e. over all values of n. The cases $n \geq N$ and $n < N$ therefore both arise, for any amplitude. As we will show next, the matrix method is nevertheless a practical, indeed efficient, method to compute couplings to closed strings.

3.3 Examples

We are now in a position to evaluate the amplitudes

$$\mathcal{A}^{(\sigma;\tilde{\sigma})} = \langle {}^\circ_\circ V^{(\sigma;\tilde{\sigma})} {}^\circ_\circ \rangle_{TBL} \,, \tag{3.15}$$

for arbitrary closed string states. We first compute the quantities,

$$I^{(\sigma;\tilde{\sigma})} = \sum_{n=0}^{\infty} (-\tilde{g})^n I_n^{(\sigma;\tilde{\sigma})} \,, \tag{3.16}$$

and then multiply by the overall normalization factor to find $\mathcal{A}^{(\sigma;\tilde{\sigma})} = \mathcal{N}'^{(\sigma;\tilde{\sigma})} I^{(\sigma;\tilde{\sigma})}$.

As the simplest example of a state at level N consider

$$\frac{1}{N} \alpha_{-N} \tilde{\alpha}_{-N} |0\rangle \,. \tag{3.17}$$

This corresponds to choosing the conjugacy class of long cycles $(12 \cdots N)$ in S_N to label both the right and left movers. In terms of partitions of N, the states correspond to the trivial partition, namely $\nu_1 = N$, $N_1 = 1$, i.e. $\sigma = \tilde{\sigma} = (N)$. For $n \geq N$, the orthogonality relation eq. (3.14) immediately gives

$$I_n^{(N;N)} = N \,; \qquad n \geq N \,. \tag{3.18}$$

[1] See for example ref. [18]

$\lambda\backslash\sigma$	(1^2)	(2)
(2)	1	1
(1^2)	1	-1

Table 1: Character table for S_2. Each row gives $\chi_\lambda(\sigma)$ for a given λ.

The case of $n < N$ is only slightly more difficult. As discussed above one must sum the characters of irreducible representations of S_N which correspond to Young frames which do not contain any columns with more than n boxes. Referring to appendix B for details, we find that there are only n Young frames which contribute a non-vanishing character, since non-vanishing contributions come from Young frames with the property that all of the boxes lie on a single hook. Furthermore, all of these have the value $\chi_\lambda(\sigma) = \pm 1$. We therefore find

$$I_n^{(N;N)} = n\,; \qquad n < N\,, \tag{3.19}$$

giving the general expression

$$I_n^{(N;N)} = \min(N, n) \qquad \forall n, N\,. \tag{3.20}$$

From this simple result, and the normalization factor $\mathcal{N}'^{(N;N)} = 2/N$, we find the amplitude

$$\mathcal{A}^{(N;N)} = \frac{2}{N}\sum_{n=0}^{\infty}(-\tilde{g})^n \min(N, n) = 2f - \frac{2}{N}\sum_{n=0}^{N-1}(N-n)(-\tilde{g})^n\,. \tag{3.21}$$

in the case of closed string states of the form (3.17).

As another example let us consider states labelled by different conjugacy classes on the two sides of the string. Here we see an enormous simplification. For $n \geq N$ the orthogonality properties of characters of the symmetric group completely kills the integral, as indicated by the delta function in eq. (3.14). Thus

$$I_n^{(\sigma;\tilde{\sigma})} = 0\,; \qquad n \geq N\,, \tag{3.22}$$

for $\sigma \neq \tilde{\sigma}$. Thus we have the remarkable result that these amplitudes only receive contributions from a finite number of terms in the perturbation series!

Let us determine these finite terms for $N = 2$. The only amplitude with different conjugacy class on each side is $\mathcal{A}^{(1^2;2)}$. Since there are no contributions with $n \geq 2$ we simply need to evaluate

$$I_n^{(1^2;2)} = \sum_{\lambda \leq n} \chi_\lambda(\sigma)\chi_\lambda(\tilde{\sigma})\,, \tag{3.23}$$

for $n = 1$. From the character table of S_2, given explicitly in table 1, we find

$$I_1^{(1^2;2)} = 1\,, \tag{3.24}$$

and therefore the amplitude is

$$\mathcal{A}^{(1^2;2)} = \mathcal{N}'^{(1^2;2)} I^{(1^2;2)} = -\sqrt{2}\,\tilde{g}\,, \tag{3.25}$$

since $\mathcal{N}'^{(1^2;2)} = \sqrt{2}$.

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$\lambda\backslash\sigma$	(3)	$(2,1)$	(1^3)
(3)	1	1	1
$(2,1)$	-1	0	2
(1^3)	1	-1	1

Table 2: Character table for S_3.

$(\sigma;\tilde{\sigma})$	$\mathcal{A}^{(\sigma;\tilde{\sigma})}$
$(1;1)$	$2[f-1]$
$(2;2)$	$[2f-2+\tilde{g}]$
$(1^2,1^2)$	$2[2f-2+\tilde{g}]$
$(1^2;2)$	$\sqrt{2}[-\tilde{g}]$
$(3;3)$	$\frac{2}{3}[3f-3+2\tilde{g}-\tilde{g}^2]$
$(2,1;2,1)$	$2[2f-2+\tilde{g}-\tilde{g}^2]$
$(1^3,1^3)$	$\frac{4}{3}[6f-6+5\tilde{g}-\tilde{g}^2]$
$(2,1;1^3)$	$\sqrt{\frac{8}{3}}[-\tilde{g}+\tilde{g}^2]$
$(2,1;3)$	$\sqrt{\frac{4}{3}}[-\tilde{g}+\tilde{g}^2]$
$(3;1^3)$	$\frac{\sqrt{8}}{3}[-\tilde{g}-\tilde{g}^2]$

Table 3: One point amplitudes of the closed strings up to level 3. The quantity in the square bracket is $I^{(\sigma;\tilde{\sigma})}$; the prefactor is $\mathcal{N}'^{(\sigma;\tilde{\sigma})}$. The function $f = 1/(1+\tilde{g})$.

Proceeding similarly we have processed the characters of S_3 given in table 2 and found all the closed string couplings up to level 3. The results are given in table 3 where, for easy reference, we include also the results from level 1,2.

3.4 Comparison with boundary states

The boundary state for the rolling tachyon background takes the form

$$|B\rangle = |B_{X^0}\rangle \otimes |B_{\vec{X}}\rangle \otimes |B_{\rm ghost}\rangle\,, \tag{3.26}$$

where $|B_{\vec{X}}\rangle$ is the usual boundary state for the spatial part of a D25 brane in bosonic string theory, $|B_{ghost}\rangle$ is the contribution from the ghosts and

$$|B_{X^0}\rangle = B^{(0;0)}|0\rangle + B^{(1;1)}\alpha^0_{-1}\tilde{\alpha}^0_{-1}|0\rangle + \frac{1}{\sqrt{2}}B^{(1,1;2)}(\alpha^0_{-1})^2\tilde{\alpha}^0_{-2}|0\rangle + \cdots\,. \tag{3.27}$$

The spatial and ghost components of the boundary state will play no role in this paper. The temporal component of the boundary state can be computed following [19, 20, 1] and gives [3, 4]

$$|B_{X^0}\rangle = \sum_j \sum_{m\geq 0} \left(\begin{pmatrix} j+m \\ 2m \end{pmatrix}\right) (-\tilde{g})^{2m} |j,m,m\rangle\rangle\,, \tag{3.28}$$

where $|j,m,m\rangle\rangle$ are the Ishibashi states, i.e. infinite sets of states built as Virasoro descendants of certain SU(2) primaries. The laborious part of finding explicit expressions for

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$(\sigma;\tilde{\sigma})$	$B^{(\sigma;\tilde{\sigma})}$
$(1;1)$	$f-2$
$(2;2)$	$f-2+\tilde{g}$
$(1^2,1^2)$	$f+2\tilde{g}$
$(1^2;2)$	$-\sqrt{2}\tilde{g}$
$(3;3)$	$f-2+\frac{4}{3}\tilde{g}-\frac{2}{3}\tilde{g}^2$
$(2,1;2,1)$	$f+\tilde{g}-2\tilde{g}^2$
$(1^3,1^3)$	$f-2+\frac{2}{3}\tilde{g}-\frac{4}{3}\tilde{g}^2$
$(2,1;1^3)$	$\sqrt{\frac{2}{3}}[\tilde{g}+2\tilde{g}^2]$
$(2,1;3)$	$\frac{2}{\sqrt{3}}[-\tilde{g}+\tilde{g}^2]$
$(3;1^3)$	$\frac{2\sqrt{2}}{3}[-\tilde{g}-\tilde{g}^2]$

Table 4: Boundary state coefficients up to level 3.

the boundary states is to work out the Ishibashi states, since these become increasingly complex at higher level. In contrast our methods get at that information quite easily.

The boundary state coefficients $B^{(\sigma;\tilde{\sigma})}$ can be represented as the one point functions

$$B^{(\sigma;\tilde{\sigma})} = \langle : V^{(\sigma;\tilde{\sigma})} : \rangle_{TBL} \,, \tag{3.29}$$

of the corresponding closed string vertex operators. It is the standard bulk normal ordering that appears in this expression, in contrast to the boundary normal ordering which, as emphasized in [3], appears in our amplitudes eq. (3.15). The two normal orderings are related by

$$: X^0(z,\bar{z}) X^0(z',\bar{z}') := {}^\circ_\circ X^0(z,\bar{z}) X^0(z',\bar{z}') {}^\circ_\circ + \log|z\bar{z}'-1| \,, \tag{3.30}$$

with the difference due to the "image" term in the disk Green's function eq. (3.2). It follows, for example, that

$$: \partial^\nu X^0(0,0) \bar{\partial}^{\nu'} X^0(0,0) : \ = {}^\circ_\circ \partial^\nu X^0(0,0) \bar{\partial}^{\nu'} X^0(0,0) {}^\circ_\circ - \frac{1}{2} \nu!(\nu-1)! \delta_{\nu\nu'} \,. \tag{3.31}$$

Taking the normalization factors into account, and recalling that the function f is the vacuum amplitude, this gives

$$B^{(N;N)} = \mathcal{A}^{(N;N)} - f \,, \tag{3.32}$$

or, from eq. (3.21),

$$B^{(N;N)} = f - \frac{2}{N} \sum_{n=0}^{N-1} (N-n)(-\tilde{g})^n \,. \tag{3.33}$$

Let us emphasize that this expression gives the boundary state for any N. The explicit computations of boundary states using Ishibashi states have been carried to level two [10], with a result that agrees with our general expression eq. (3.32). Extending the boundary computations to higher N is however very tedious and in general not very practical.

The relation eq. (3.30) between normal orderings can be extended to more complex operators. The general result has the same form as Wick's rule, except that contraction terms here appear only for operators with an identical number of derivatives. The two normal orderings are thus equivalent when $\sigma, \tilde{\sigma}$ have no cycles that are of the same length. The awkward numerical coefficient $-\frac{1}{2}\nu!(\nu-1)!\delta_{\nu\nu'}$ that comes with each contraction conspires with the overall normalization $\mathcal{N}'^{(\sigma;\tilde{\sigma})}$ to give simple combinatorial factors when relating $B^{(\sigma;\tilde{\sigma})}$ and $\mathcal{A}^{(\sigma;\tilde{\sigma})}$. For example, if only one type of cycle appears

$$B^{(\nu^N;\nu^N)} = \mathcal{A}^{(\nu^N;\nu^N)} - N\mathcal{A}^{(\nu^{N-1};\nu^{N-1})} + \frac{N(N-1)}{2}\mathcal{A}^{(\nu^{N-2};\nu^{N-2})} + \cdots . \tag{3.34}$$

We have used the contraction rules to compute boundary state coefficients from our matrix amplitudes for all states up to level three. The results are given in table 4. As a check we have carried the boundary state computations one level higher than [10], to level three, and verified agreement of all terms. This gives us great confidence that the two methods really are equivalent. It also shows that the matrix method is by far the most convenient.

4. Bulk correlators

In this section we consider amplitudes of the form

$$\mathcal{A}^{\Pi \exp(-n_k X^0)} = \left\langle : \prod_{k=1}^{m} e^{-n_k X^0(z_k,\bar{z}_k)} : \right\rangle_{TBL} , \tag{4.1}$$

where the z_k are in the interior of the unit disk. These amplitudes are the building blocks of general bulk correlators in the background of the rolling tachyon. They were previously considered in [4] where explicit results were presented for $m \leq 2$. We present these here both to reiterate our general theme that all correlation functions in this theory are matrix integrals and to demonstrate the ease with which the resulting integrals can be evaluated, even in the general case.

Although it is not strictly necessary for our methods to work, we will enforce momentum conservation. Then the only contribution to the amplitude is at nth order in perturbation theory where $n = \sum n_k$; and so, up to an overall factor of $(-2\pi g)^n$, the entire amplitude reduces to

$$\mathcal{A}_n^{\Pi \exp(-n_k X^0)} = \frac{1}{n!}\left\langle \prod_{k=1}^{m} e^{-n_k X^0(z_k,\bar{z}_k)} \prod_{i=1}^{n} \int \frac{dt_i}{2\pi} e^{X^0(w_i,\bar{w}_i)} \right\rangle , \tag{4.2}$$

where the w_i are situated on the boundary. Straightforward calculation of the contractions, using the Green's function eq. (3.2), gives

$$\begin{aligned}\mathcal{A}_n^{\Pi \exp(-n_k X^0)} &= \frac{1}{n!}\prod_{k=1}^{m} |z_k\bar{z}_k - 1|^{n_k^2/2} \prod_{k<l} |z_k - z_l|^{n_k n_l} |z_k\bar{z}_l - 1|^{n_k n_l} \times \\ &\quad \times \prod_{i=1}^{n} \int \frac{dt_i}{2\pi} \prod_{i<j} |e^{it_i} - e^{it_j}|^2 \prod_{k=1}^{m}\prod_{i=1}^{n} |z_k - w_i|^{-2n_k} ,\end{aligned} \tag{4.3}$$

where we have used $\bar{w} = 1/w = e^{-it}$ on the boundary of the unit disk. The last term in the integrand can be rewritten

$$\prod_{k=1}^{m}\prod_{i=1}^{n}|z_k - w_i|^{-2n_k} = \prod_{k,i}(1 - z_k y_i)^{-n_k}(1 - \bar{z}_k \bar{y}_i)^{-n_k}\,, \tag{4.4}$$

where we have defined $y_i = 1/w_i$ and used the fact that $w_i\bar{w}_i = 1$. But the product

$$h(z_k) \equiv \prod_i (1 - z_k y_i)^{-1} = \det(1 - z_k U^\dagger)^{-1}\,, \tag{4.5}$$

is expressed directly in terms of matrices; and so

$$\begin{aligned}\mathcal{A}_n^{\Pi\exp(-n_k X^0)} &= \prod_{k=1}^{m}|z_k\bar{z}_k - 1|^{n_k^2/2}\prod_{k<l}|z_k - z_l|^{n_k n_l}|z_k\bar{z}_l - 1|^{n_k n_l}\times\\ &\quad\times\int\frac{dU}{\mathrm{vol}(\mathrm{U}(n))}\prod_k h(z_k)^{n_k}h(z_k)^{\dagger n_k}\,,\end{aligned} \tag{4.6}$$

which is the advertized formula for this entire class of correlators, written in terms of unitary matrix integrals.

In order to evaluate the integral eq. (4.6) first recast $h(z_k)^{n_k}$ in a more convenient form by defining a new set of variables

$$\{Z_K\} = (\overbrace{z_1, z_1\cdots, z_1}^{n_1}; \overbrace{z_2, z_2, \cdots, z_2}^{n_2}\cdots; \overbrace{z_m, z_m, \cdots, z_m}^{n_m})\,, \tag{4.7}$$

where the index $K \in (1, n)$ with $n = \sum_i n_i$. This enables us to write

$$\prod_{k=1}^{m}h(z_k)^{n_k} = \prod_{k,i}(1 - z_k y_i)^{-n_k} = \prod_{K,i}(1 - Z_K y_i)^{-1} = \prod_K h(Z_K)\,. \tag{4.8}$$

Now, the expression eq. (4.5) is in fact the textbook definition of the generating function for what are known as the complete symmetric polynomials — see appendix C. These are fundamental objects in the theory of symmetric functions since, among other things, they provide a basis for the ring of symmetric polynomials. Also of central importance is the Cauchy identity

$$\prod_{K,i}(1 - Z_K y_i)^{-1} = \sum_\lambda s_\lambda(Z)s_\lambda(y)\,, \tag{4.9}$$

where $s_\lambda(x)$ are the Schur functions for the abstract variables $x = \{x_i\}$, labelled by partitions (Young frames) λ. In the case we are considering $y_i = e^{-it_i}$ are the eigenvalues of the matrix $U^\dagger$ and therefore the Schur function $s_\lambda(y)$ is known to be equivalent to the character of $U^\dagger$ in the irreducible representation labelled by the partition λ *i.e.*, $s_\lambda(w) = \chi_\lambda(U^\dagger)$. We will not need the explicit form of the Schur functions for the variable $Z = \{Z_K\}$. We may now write

$$\frac{1}{\mathrm{vol}(\mathrm{U}(n))}\int dU\prod_K h(Z_K)h(Z_K)^\dagger = \sum_{\lambda\lambda'}s_\lambda(Z)s_{\lambda'}(\bar{Z})\frac{1}{\mathrm{vol}(\mathrm{U}(n))}\int dU\chi_\lambda(U)\bar{\chi}_{\lambda'}(U)$$

$$= \sum_{\lambda\lambda'} s_\lambda(Z) s_{\lambda'}(\bar{Z}) \delta_{\lambda\lambda'}$$

$$= \sum_{\lambda} s_\lambda(Z) s_\lambda(\bar{Z}) \,, \tag{4.10}$$

where we have once again evaluated the integral over $\mathrm{U}(n)$ using the orthogonality of group characters. Instead of computing the remaining sum term by term we can use the Cauchy identity in reverse to obtain

$$\sum_\lambda s_\lambda(Z) s_\lambda(\bar{Z}) = \prod_{I,J=1}^{n} (1 - Z_I \bar{Z}_J)^{-1} = \prod_{i,j=1}^{m} (1 - z_i \bar{z}_j)^{-n_i n_j} \,. \tag{4.11}$$

Assembling eqs. (4.6), (4.10) and (4.11) we obtain the general form of the correlator

$$\mathcal{A}_n^{\Pi \exp(-n_k X^0)} = \prod_{k=1}^{m} |z_k \bar{z}_k - 1|^{-n_k^2/2} \prod_{i<j}^{m} |z_i - z_j|^{n_i n_j} \prod_{i<j}^{m} |z_i \bar{z}_j - 1|^{-n_i n_j} \,. \tag{4.12}$$

This generalizes the result of [4] to include all m-point functions of bulk tachyons for $m > 2$. Note that we have made no recourse here to the SU(2) current algebra, nor to contour integration.

5. Boundary correlators

For completeness we point out that correlators involving insertions of vertex operators on the boundary of the disk can also be written as matrix integrals. In fact this follows almost trivially from the previous section. The general boundary correlator is

$$\tilde{\mathcal{A}}_n^{\Pi \exp(-n_k X^0)} = \frac{1}{n!} \left\langle \prod_{k=1}^{m} {}^\circ_\circ e^{-n_k X^0(z_k, \bar{z}_k)} {}^\circ_\circ \prod_{i=1}^{n} \int \frac{dt_i}{2\pi} e^{X^0(w_i, \bar{w}_i)} \right\rangle , \tag{5.1}$$

where now the z_k are points on the boundary of the disk. The contractions then give

$$\tilde{\mathcal{A}}_n^{\Pi \exp(-n_k X^0)} = \frac{1}{n!} \prod_{k<l}^{m} |z_k - z_l|^{2 n_k n_l} \prod_{i=1}^{n} \int \frac{dt_i}{2\pi} \prod_{i<j} |e^{it_i} - e^{it_j}|^2 \prod_{k=1}^{m} \prod_{i=1}^{n} |z_k - w_i|^{-2n_k} \,. \tag{5.2}$$

The simplifications relative to eq. (4.3) are due to $|z| = 1$. Another difference is that we now use boundary normal ordering to regulate the vertex operators. We find

$$\tilde{\mathcal{A}}_n^{\Pi \exp(-n_k X^0)} = \prod_{k<l}^{m} |z_k - z_l|^{2 n_l n_k} \int \frac{dU}{\mathrm{vol}(\mathrm{U}(n))} \prod_k h(z_k)^{n_k} h(z_k)^{\dagger n_k} \,, \tag{5.3}$$

with the same definitions as in section five above. Proceeding as before the boundary amplitude becomes

$$\tilde{\mathcal{A}}_n^{\Pi \exp(-n_k X^0)} = \prod_{k=1}^{m} |z_k \bar{z}_k - 1|^{-n_k^2} \,. \tag{5.4}$$

This expression is clearly divergent since $z_k \bar{z}_k = 1$ for points on the boundary. Dealing with this divergence directly in timelike boundary Liouville theory is difficult and is perhaps best dealt with by reverting to analytic continuation from the spacelike boundary Liouville theory [21] as advocated in [4]. We are hopeful however that the matrix integral perspective may circumvent this indirect approach.

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6. Rolling tachyons and QCD_2

In section three we observed that the one point functions of massive string states in the background of the rolling tachyon take the form,

$$\langle {}^{\circ}_{\circ} V^{(\sigma;\tilde{\sigma})} {}^{\circ}_{\circ} \rangle_{TBL} = \mathcal{N}'^{(\sigma;\tilde{\sigma})} \sum_{n=0}^{\infty} (-\tilde{g})^n I_n^{(\sigma;\tilde{\sigma})} \,, \tag{6.1}$$

where the summand is given by the U(n) integral,

$$I_n^{(\sigma;\tilde{\sigma})} = \frac{1}{\mathrm{vol}(\mathrm{U}(n))} \int dU \prod_{i,\tilde{i}} [\mathrm{Tr}(U^{\nu_i})]^{N_i} [\mathrm{Tr}(U^{\dagger \tilde{\nu}_{\tilde{i}}})]^{\tilde{N}_{\tilde{i}}} \,. \tag{6.2}$$

As we already noted, this is *exactly* the correlation function of a pair of Wilson loops in two dimensional U(n) gauge theory in the limit where the area of the two dimensional manifold vanishes [15, 16]. The precise correspondence is somewhat unusual since from eq. (6.1) the full correlator in the rolling tachyon background is given by the sum over Wilson loop correlators in different gauge theories! In other words,

$$\langle {}^{\circ}_{\circ} V^{(\sigma;\tilde{\sigma})} {}^{\circ}_{\circ} \rangle_{TBL} = \sum_{n=0}^{\infty} (-\tilde{g})^n \langle W_n(\sigma) \bar{W}_n(\tilde{\sigma}) \rangle_{QCD_2} \,. \tag{6.3}$$

where $W_n(\sigma)$ is a Wilson loop operator in two dimensional U(n) gauge theory.

Since correlators involving other vertex operators in the rolling tachyon background similarly reduce to matrix integrals we make the following proposal: given a set of vertex operators $V_i(z, \bar{z})$ then general correlation functions in the timelike boundary Liouville theory can be expressed as

$$\left\langle \prod_i V_i(z, \bar{z}) \right\rangle_{TBL} = \sum_{n=0}^{\infty} (-\tilde{g})^n \left\langle \prod_i Q_i^n(z; \alpha) \bar{Q}_i^n(\bar{z}; \tilde{\alpha}) \right\rangle_{QCD_2} \,, \tag{6.4}$$

where $Q_i^n(z; \alpha)$ is a gauge invariant operator in U(n) Yang-Mills with α representing how this operator depends on the details of the Liouville vertex operators $V_i(z, \bar{z})$. As an empirical formula (6.4) is certainly valid in the cases we have considered in this paper. The examples we have uncovered thus far can be usefully organized as

$$\begin{aligned} \prod_i \left[\partial^{\nu_i} X^0(0)\right]^{N_i} &\leftrightarrow \prod_i \left[\mathrm{Tr}(U^{\dagger \nu_i})\right]^{N_i} \leftrightarrow \sum_\lambda \chi_\lambda(\sigma) \chi_\lambda(U^\dagger) \,, \\ \exp(-X^0(z)) &\leftrightarrow \det(1 - zU^\dagger)^{-1} \leftrightarrow \sum_\lambda s_\lambda(z) \chi_\lambda(U^\dagger) \,, \end{aligned} \tag{6.5}$$

where we have focused on the holomorphic parts of the vertex operators in question and the implied correspondence holds at the level of correlation functions. Viewed this way it is clear that the characters of the unitary group $\chi_\lambda(U)$ are playing the role of a complete set of functions into which all the operators can be expanded. The expansion coefficients on the other hand are dependent on the specific vertex operator. It would be interesting to uncover the general rule for associating Liouville vertex operators with gauge invariant operators of QCD_2.

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It is perhaps useful to explain in more detail how our envisioned correspondence is realized. First, we recall the interpretation of QCD_2 as a string theory [15]. The fundamental idea is that the partition function can be understood as the sum over all maps of a two dimensional world sheet onto a two dimensional target space of fixed topology. The maps in question are usually referred to as covering maps and have an associated winding number N indicating that the map covers the target space N times. The physical observables of any gauge theory are the gauge invariant operators, a particular example of which is the Wilson loop,

$$W(\sigma) = \mathrm{Tr}\left(e^{\oint_\sigma A\cdot dx}\right) \equiv \mathrm{Tr}(U_\sigma)\,. \tag{6.6}$$

where the matrix U_σ represents the holonomy of the gauge field as one traverses the path defined by σ. In the stringy realization of QCD_2 a Wilson loop is an S^1 inside of the target space onto which the boundary of a worldsheet is mapped. Since the target space will in general have a finite number of punctures the map will have branch points. As one traverses the S^1 of the target space one will encounter branch points where the different sheets of the map meet. The path is thus determined by specifying how the sheets are permuted into each other. Therefore, in addition to the winding number N, we also label each Wilson loop by an element $\sigma \in S_N$. Here σ should be thought of as defining the path by specifying how the sheets are permuted as one traverses the Wilson loop. In string theory language the natural set of gauge invariant observables are [16],

$$\mathrm{Tr}(U_\sigma) = \prod_i \left[\mathrm{Tr}(U^{\nu_i})\right]^{N_i} \tag{6.7}$$

where on the right hand side U is taken to be in the defining representation of $\mathrm{U}(n)$ and $(\nu_i; N_i)$ specify a partition of N. With this suggestive notation it is clear how we should translate between the QCD_2 language and that of the rolling tachyon: we identify, respectively, N and σ with the level number and oscillator structure of the massive closed string whose one point function we are calculating in the rolling tachyon background. The interpretation is that these one point functions of timelike boundary Liouville theory are encoding information about how Riemann surfaces with boundaries can be mapped into Riemann surfaces with a fixed number of punctures. Note that the correspondence we are suggesting is strictly valid only in the limit of vanishing area of the target space. Thus, intuitively, the information contained in the Liouville theory should only be topological. Also, since we must include all values of n it is not completely clear how the geometric interpretation should manifest itself since the above stringy interpretation of QCD_2 is usually associated with a large-n limit. We leave investigation of these interesting issues to future work.

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A. Schur-Weyl duality

The purpose of this appendix is to provide an introduction to Schur-Weyl duality, used repeatedly in the main text. For more extensive discussion and formal proofs we refer to any one of a large number of standard texts on the subject. The discussion presented here largely follows ref. [22].

The basic observation of Schur and Weyl is that there is a deep connection between the representation theory of the symmetric group S_N and that of the unitary group $\mathrm{U}(n)$. The defining representation of $\mathrm{U}(n)$ is given by $n \times n$ unitary matrices which act on an n-dimensional complex vector space V which is referred to as the carrier space of the representation. One now considers the N-fold tensor product space $\mathcal{V}_N = V \otimes V \otimes \cdots \otimes V$ which is acted on by the group $H = \mathrm{U}(n) \times S_N$. Here S_N is the symmetric group of N objects and acts by permuting the N factors appearing in $\mathcal{V}_N$. Given an arbitary element $h = \sigma \times U \in H$ where $\sigma \in S_N$ and $U \in \mathrm{U}(n)$ then h in its defining representation $D(h)$ acts on the basis vectors v_{i_k} in the product space as

$$v_{i_1} v_{i_2} \cdots v_{i_N} \stackrel{D(h)}{\to} U_{i_1\sigma(j_1)} U_{i_2\sigma(j_2)} \cdots U_{i_N\sigma(j_N)} v_{j_1} v_{j_2} \cdots v_{j_N} \equiv U^{\sigma}_{(i)(j)} v_{j_1} v_{j_2} \cdots v_{j_N}\,, \tag{A.1}$$

where the notation $(i) \equiv (i_1, i_2 \cdots i_N)$ has been used and the superscript σ indicates that the permutation is to be implemented on the set of integers (j). Clearly the actions of the abstract group elements σ and U commute with each other. From the point of view of the group $\mathrm{U}(n)$ one would like to understand what are the invariant subspaces. Put differently, the action of the unitary group on the tensor product space is in general reducible and one would like to know which irreducible representations of $\mathrm{U}(n)$ appear in the decomposition. Schur-Weyl duality answers this question.

To see how this comes about note that the irredicble representations of S_N correspond to subspaces of $\mathcal{V}_N$ that are invariant (under S_N) . Now, the irreducible representations of S_N are in a one to one correspondence with the partitions of N into integers which, in turn, are nicely represented in terms of Young frames. Schur-Weyl duality is the statement that the $\mathrm{U}(n)$ invariant subspaces of $\mathcal{V}_N$ are in a one to one correspondence with the S_N invariant subspaces. Operationally this means the irreducible representations of $\mathrm{U}(n)$ appearing in the decomposition are precisely those corresponding to the Young frames classifying the irreducible representations of S_N, with each $\mathrm{U}(n)$ representation appearing precisely once.

It is perhaps instructive to see this in a simple example [22]. Consider the case $N = 2$ so that the elements of $\mathcal{V}_2$ are given by the second rank tensor F_{ij}. If we represent the elements of S_2 by (e, s) then we have $eF_{ij} = F_{ij}$ and $sF_{ij} = F_{ji}$. The irreducible representations of S_2 are summarized by the Young frames depicted in figure (1).

Now, the operators $e + s$ and $e - s$ (known as Young operators) act on the F_{ij} as projections on to the symmetric and antisymmetric pieces of F_{ij}, respectively, which are clearly the S_2 invariant subspaces of $\mathcal{V}_2$. This is also clearly what one expects as the invariant subspaces under $\mathrm{U}(n)$. Since both $e + s$ and $e - s$ commute with the action of $\mathrm{U}(n) \times \mathrm{U}(n)$ they also project out invariant subspaces of the carrier space $\mathcal{V}_2$. The irreducible representations of $\mathrm{U}(n)$ which occur are then also completely summarized in figure (1).

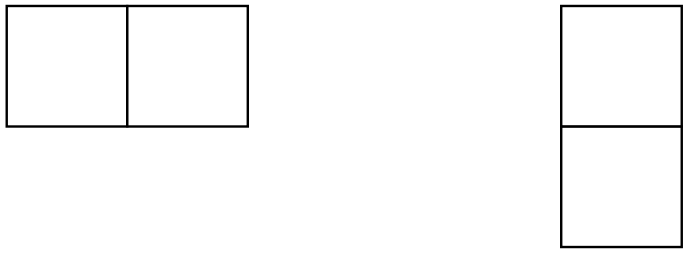

Figure 1: Young frames for S_2.

For general S_N we can similarly think of the defining representation $D(h)$ of $h = \sigma \times U$ as a tensor representation which in general is reducible and decomposes according to the irreducible representations of S_N as

$$D(h) = \sum_{\lambda} S_\lambda(\sigma) \otimes T_\lambda(U)\,, \tag{A.2}$$

where $S_\lambda(\sigma)$, $T_\lambda(U)$ are the irreducible representations of σ and U appearing in the decomposition. Again the index λ should be thought of as indexing the Young frames defining the irreducible representations of S_N. It is clear from this tensorial point of view that the entries on the right hand side of eq. (A.2) corresponding to diagrams with columns of height more than n actually vanish as representations of $\mathrm{U}(n)$. Note that we keep columns with height exactly n because we are considering $\mathrm{U}(n)$, rather than $\mathrm{SU}(n)$. The reasoning appealed to here is more or less the standard one leading to the use of Young frames to label $\mathrm{U}(n)$ representation. Of course a more rigourous treatment is possible, see [23].

Our main interest is the consequences of eq. (A.2) for the characters. Taking the trace, we find

$$\mathrm{Tr}(D(h)) = \sum_{\lambda} \chi_\lambda(\sigma)\chi_\lambda(U)\,, \tag{A.3}$$

since the character of a direct product is the product of the characters. Note that, since the character is the same for all elements of a conjugacy class, we can now think of the σ in the argument of χ_λ as a conjugace class, rather than a group element. The left hand side can be more explicit by considering the expression eq. (A.1) for the defining representation $D(h)$ and directly taking the trace of the operator $U^{\sigma}_{(i)(j)}$, by setting $(i) = (j)$ and summing over (j). We find

$$\mathrm{Tr}U^{\sigma}_{(i)(j)} = \sum_{(j)} U^{\sigma}_{(j)(j)} = \prod_k [\mathrm{Tr}(U^{\nu_k})]^{N_k}\,, \tag{A.4}$$

where the permutation σ contains N_1 cycles of length ν_1, N_2 cycles of length ν_2 etc. For example if $\sigma = (1^N)$ *i.e.*, the identity permutation $\sigma = e$, then clearly

$$\mathrm{Tr}U^{(1^N)}_{(i)(j)} = \sum_{j_1}\sum_{j_2}\cdots\sum_{j_N} U_{j_1 j_1} U_{j_2 j_2}\cdots U_{j_N j_N} = [\mathrm{Tr}U]^N\,, \tag{A.5}$$

and likewise, if $\sigma = (N)$, then

$$\mathrm{Tr}U^{(N)}_{(i)(j)} = \sum_{j_1}\sum_{j_2}\cdots\sum_{j_N} U_{j_1 j_2} U_{j_2 j_3}\cdots U_{j_N j_1} = \mathrm{Tr}(U^N)\,. \tag{A.6}$$

Generally, we find

$$\prod_k [\mathrm{Tr}(U^{\nu_k})]^{N_k} = \sum_\lambda \chi_\lambda(\sigma)\chi_\lambda(U)\,, \tag{A.7}$$

which is the formula used extensively in the main body of the paper, to simplify integrals .

B. The Murnaghan-Nakayama rule

Our algebraic algorithm for determining the couplings to closed strings ultimately relies on the computation of characters of the symmetric group. In this section we review a simple graphical technique for doing this, known as the Murnaghan-Nakayama rule. (For a derivation see [24].) With this rule in hand the industrious reader may verify the explicit results quoted in the main text, or find more general ones.

The goal is to calculate the characters $\chi_\lambda(\sigma) = \mathrm{Tr}_\lambda(\sigma)$. We first recall two fundamental facts about the symmetric group. First, the conjugacy classes σ of S_N, which are defined by specifying a cycle structure, are in a one to one correspondence with the partitions of N. Second, the irreducible representations λ of S_N are also in a one to one correspondence with the partitions of N. Thus the input required to calculate the character of a given conjugacy class in a given representation is simply a pair of partitions of N. In the following we will use two techniques to encode a partition of N. First, we will identify a given representation λ with a Young frame as follows. We can always write a partition as $N = \lambda_1 + \lambda_2 + \cdots \lambda_k$ where by convention we take $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$. We then construct the Young frame by drawing a row of boxes of length λ_1. Immediately under this (and aligned on the left) we draw a row of boxes of length λ_2. We continue this until we have a row of boxes for each of the λ_i. Each frame will (by definition) contain N boxes. Each possible shape corresponds to a different partition of N and thus a specific irreducible representation λ. Next, a given conjugacy class σ is specified by its cycle structure and we can therefore label a conjugacy class by specifying the lengths of it's cycles *i.e.,* $\sigma = (\mu_1, \mu_2, \cdots \mu_k)$ where $\sum_i \mu_i = N$. Note that the notation used in this rule differ from $\sigma = (\nu_1^{N_1}, \cdots, \nu_k^{N_k})$ used previously by introducing a distinct label for each of several cycles having the same length. With this input the character of σ in the representation λ can be calculated with the following rule:

- Draw the Young frame correponding to λ.
- Fill in the boxes with μ_1 1's, μ_2 2's, μ_3 $3's$, μ_4 4's etc. so that
  1. Each set of numbers forms a continous hook pattern. By this we mean an uninterrupted vertical line, followed by a horizontal line to the right; or a horizontal line to the right, follwed by a vertical line up.
  2. The numbers are weakly increasing from left to right and top to bottom.
- Each table is assigned the number $H = (-1)^{\sum(h_i-1)}$ where h_i is the height of each hook, i.e. the number of boxes in the vertical part of the hook.

Figure 2: A demonstration of the Murnaghan-Nakayama Rule.

(a) (b)

Figure 3: (a) A frame which allows a continuous hook, and thus gives a non-vanishing character. (b) A frame which does not allow any continuous hooks, and thus gives a vanishing character.

- Repeat the above procedure for all possible ways of covering the Young frame with hooks according to the above rules.
- The character $\chi_\lambda(\sigma)$ is given by the sum of all the H values for each covering.

The computations using the Murnaghan-Nakayama rule are in fact simpler than it may appear at first. Let us consider an explicit example. Take the representation $\lambda = (4,3,2)$ of S_9 and let us evaluate its character on the group element $\sigma = (12)(345)(678)(9)$ which belongs to the conjugacy class which we can write as $\sigma = (2,3,3,1)$, where the entries indicate the cycle lengths. Notice $2+3+3+1 = 9$ and also $4+3+2 = 9$; so these are partitions of 9. The procedure for computing the character $\chi_{(4,3,2)}(2,3,3,1)$ is then spelled out in figure (2). Each table has the shape corresponding to the representation $(4,3,2)$ and there are only two possible coverings with numbers consistent with the rules given above. Each of these tables has $H = 1$; so $\chi_{(4,3,2)}(2,3,3,1) = 2$.

As another class of examples, consider the conjugacy class of long cycles in S_N *i.e.*, $\sigma = (N)$ in our present notation. There are lot's of Young frames with N boxes; but we need to cover a given frame with a hook of N 1's and this leaves only single hook diagrams. For example figure (3a) can accomodate a hook, but figure (3b) cannot. For each frame allowing a hook there is obviously only one way to cover the frame so, whatever the height, $H = (-)^{h_1-1} = \pm 1$. Thus, as claimed in the main text, $\chi_\lambda(N) = \pm 1$ for all the representations with non-zero character on the long cycle. If we restrict to representations with no more than n boxes in a column there are precisely n such frames and so $\sum_{\lambda \leq n} \chi^2_\lambda(N) = n$.

Explicit computations using the Murnaghan-Nakayama rule can obviously become quite laborious for complex representations. The point we want to make here is simply that the computation of amplitudes is completely algebraic. Indeed, for practical computations

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at high level there even exists a MAPLE code [25] which automates the determination of the characters, and thus the amplitudes.

C. Schur functions and the Cauchy identity

This appendix will attempt to summarize some basic facts and definitions from the theory of symmetric functions that are useful for evaluating various integrals in the main sections of the paper. For proofs and derivations there is a clear discussion of this enormous subject in ref. [24].

Consider a partition λ of the form $\lambda_1 + \cdots + \lambda_n = N$ where, in the present context, some of the λ_i may vanish. We can then define a symmetric polynomial in n variables and degree N as

$$m_\lambda = \sum_{\sigma \in S_n} x_1^{\sigma(\lambda_1)} x_2^{\sigma(\lambda_2)} \cdots x_n^{\sigma(\lambda_n)} . \tag{C.1}$$

For $N \leq n$ the special case $\lambda_i = 1$ $(i \leq N)$ and $\lambda_i = 0$ $(N < i \leq n)$ gives rise to what are known as the elementary symmetric functions in n variables

$$e_N = \sum_{1 \leq i_1 < i_2 \cdots < i_N \leq n} x_{i_1} x_{i_2} \cdots x_{i_N} , \tag{C.2}$$

and can be represented by the generating series

$$e(t) = \sum_{N \geq 0} e_N t^N = \prod_{i=1}^{n} (1 + t x_i) . \tag{C.3}$$

Another class of symmetric functions are the complete symmetric functions h_k. They are defined as

$$h_N = \sum_{1 \leq i_1 \leq i_2 \cdots \leq i_N \leq n} x_{i_1} x_{i_2} \cdots x_{i_N} , \tag{C.4}$$

and they have the generating function

$$h(t) = \sum_{N \geq 0} h_N t^N = \prod_{i=1}^{n} (1 - t x_i)^{-1} . \tag{C.5}$$

Note that $e(t)h(-t) = 1$.

It is also useful to consider the anti-symmetric functions defined by

$$a_\lambda = \sum_{\sigma \in S_n} \epsilon(\sigma) x_1^{\sigma(\lambda_1)} x_2^{\sigma(\lambda_2)} \cdots x_n^{\sigma(\lambda_n)} , \tag{C.6}$$

where $\epsilon(\sigma)$ is the sign of the permutation σ. These functions are completely antisymmetric under the interchange of any two of the variables x_i. The sets of symmetric and anti-symmetric functions are isomorphic to each other. The isomorphism can be realized concretely as multiplication by the Vandermonde determinant

$$\Delta(x) = \det(x_i^{n-j}) = \prod_{1=i<j}^{n} (x_i - x_j) , \tag{C.7}$$

where the notation x_i^{n-j} is understood to indicate the (i, j) entry of an $n \times n$ matrix.

With the above definitions we now come to the Schur functions. These are a very general class of symmetric functions which contain as special cases both the elementary and complete symmetric functions above. A Schur function $s_\lambda(x)$ for some abstract variables x_i is specified by a partition λ (of the same form as above) as

$$s_\lambda(x) = \frac{\det(x_i^{\lambda_j+n-j})}{\det(x_i^{n-j})} . \tag{C.8}$$

The Schur functions are symmetric polynomials of degree N. For example, if $\lambda_1 = N$ and $\lambda_i = 0, i \in (2, n)$ then $s_\lambda(x)$ reduces to the complete symmetric function h_N. When $\lambda_i = 1, i \in (1, n)$ (C.8) gives the elementary symmetric fucntion e_N. For the purposes of this paper the Schur functions are useful because of their connection to the characters of the unitary group. When the variables $x_i = e^{it_i}$ it turns out that $s_\lambda(x) = \chi_\lambda(U)$, where U is the unitary matrix for which the e^{it_i} are the eigenvalues.

We now state, without proof, the Cauchy identity. This is a remarkable identity which relates *products* of complete symmetric functions to Schur polynomials. Consider a set of generating functions for complete symmetric functions $h(z_l)$, $l \in (1, m)$. Then the product over this set can be written in terms of Schur functions as

$$\prod_{l=1}^{m} h(z_l) = \prod_{l=1}^{m}\prod_{i=1}^{n}(1 - z_l x_i)^{-1} = \sum_\lambda s_\lambda(z) s_\lambda(x) . \tag{C.9}$$

For a proof of this statement see ref. [24]. This relation is very general and holds for any two abstract sets of variables $z = \{z_l\}$ and $x = \{x_i\}$. We will only need the case $n = m$ where the representations λ are general frames with n boxes.

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